10. Length of a Curve and Curvature

In this lecture, we will discuss

- Length of a Curve
  - Definition of the Length of a Path
  - Length of a Curve (equal to the length of a special class of paths, called smooth paths.)
  - Arc Length Function
- Curvature

Remark. For the homework and exam, we mainly focus on the part about length of a curve (No homework questions and exam questions on the curvature).

Length of a Curve

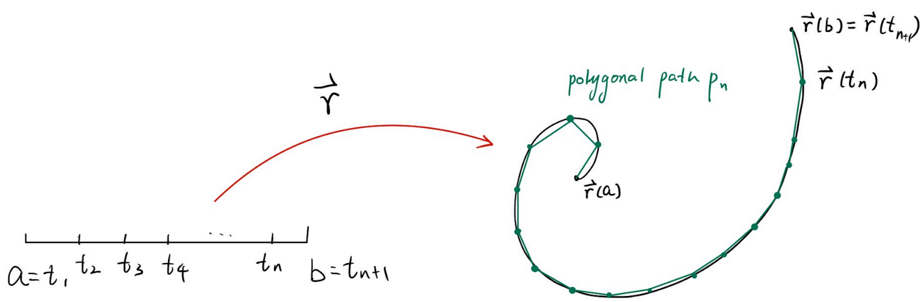
The following discussion helps us understand why we use the integral to calculate the length of a path.

Let $\mathbf{r}(t) = (x(t), y(t)) : [a, b] \to \mathbb{R}^2$ be a differentiable path in the xy-plane.

Question. How can we compute its length?

We following the steps:

- Divide the interval $[a, b]$ into n subintervals $[a = t_1, t_2], [t_2, t_3], \ldots, [t_n, t_{n+1} = b]$ and join the corresponding points on the curve $\mathbf{r}(t_1), \mathbf{r}(t_2), \ldots, \mathbf{r}(t_{n+1})$ on $\mathbf{r}$ with straight-line segments $r_1, r_2, \ldots, r_n$, forming a polygonal path p_n.
- The lengths of $r_1, \ldots, r_n$ are $\ell(r_1) = \|\mathbf{r}(t_2) - \mathbf{r}(t_1)\|, \ldots, \ell(c_n) = \|\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)\|$, and the total length of p_n is $\ell(p_n) = \ell(c_1) + \cdots + \ell(c_n)$.



- Keep increasing n, we obtain polygonal paths p_n that approximate the path $\mathbf{r}$ better and better.
- The length $\ell(\mathbf{r})$ of $\mathbf{r}$ is defined as the limit of lengths of those polygonal paths as $n \to \infty$:

$$\begin{aligned}
\mathbf{r}'(t_i) &= x'(t_i)\mathbf{i} + y'(t_i)\mathbf{j} \\
&\approx \frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}\mathbf{i} + \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i}\mathbf{j} \\
&= \frac{(x(t_{i+1}) + y(t_{i+1}))\mathbf{i} - (x(t_i) + y(t_i))\mathbf{j}}{t_{i+1} - t_i} = \frac{\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)}{t_{i+1} - t_i}
\end{aligned}$$

Roughly speaking, we have

$$\mathbf{r}'(t_i) \approx \frac{\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)}{t_{i+1} - t_i}$$

- Thus the length of the i th segment, l_i, is approximately equal to

$$\ell(r_i) = \|\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)\| \approx \|\mathbf{r}'(t_i)\|(t_{i+1} - t_i) = \|\mathbf{r}'(t_i)\|\Delta t_i,$$

where $\Delta t_i = t_{i+1} - t_i$.

- Consequently, the length of the polygonal path p_n

$$\ell(p_n) = \sum_{i=1}^{n} \ell(\vec{r}_i) \approx \sum_{i=1}^{n} \|\mathbf{r}'(t_i)\|\Delta t_i$$

approximates the length of the curve. Now compute the limit as $n \to \infty$.

- By definition, the limit $\lim_{n\to\infty} \ell(p_n)$ on the left side is equal to the length $\ell(\mathbf{r})$ of the path $\mathbf{r}$.
- The limit on the right side is the definite integral of the function $\|\mathbf{r}'(t)\|$ over the interval $[a, b]$.
- This argument justifies the following definition.

Definition. Length of a Path
Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) be a path such that its derivative is continuous. The length $\ell(\mathbf{r})$ of $\mathbf{r}$ is given by

$$\ell(\mathbf{r}) = \int_a^b \|\mathbf{r}'(t)\| dt.$$

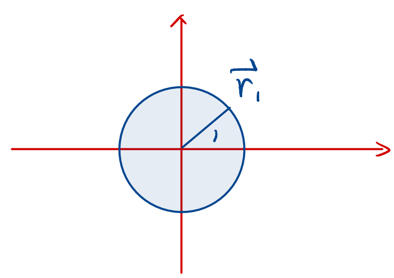
Remark. Note so far we defined the length of a path, not a curve in the title of the lecture. The following example explains why we distinguish the definitions of a path and a curve. The definition of a curve is equal to the length of a special class of paths, called smooth paths. See the definitions after this example.

Example 1.

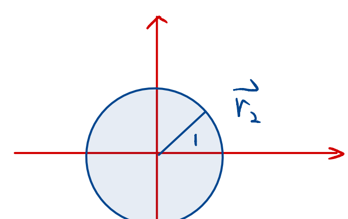
Compute the length of the paths $\mathbf{r}_1(t) = (\cos t, \sin t), t \in [0, 2\pi]$ and $\mathbf{r}_2(t) = (\cos 3t, \sin 3t), t \in [0, 2\pi]$ in $\mathbb{R}^2$. ANS:

By definition

$$\ell(\vec{r}_1) = \int_0^{2\pi} \|\vec{r}_1'(t)\| dt$$
$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$
$$= \int_0^{2\pi} 1\, dt = t \Big|_0^{2\pi} = 2\pi$$



$$\ell(\vec{r}_2) = \int_0^{2\pi} \|\vec{r}_2'(t)\| dt$$
$$= \int_0^{2\pi} \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2}\, dt$$
$$= \int_0^{2\pi} 3 dt = 3t \Big|_0^{2\pi} = 6\pi$$



Both $\vec{r}_1$ & $\vec{r}_2$ represent the same curve, which is the circle of radius 1.
But only the first parametrization gives its length

The path $\vec{r}_2$ travels the circle 3 times, thus the length of the path is 3 times the length of the circle.

Therefore, if we want to measure the length of a curve, we have to exclude some parametrizations, such as $\mathbf{r}_2$ in **Example 1**.

To do this, we introduce the following notion of a smooth path.

Definition Smooth Path (Smooth Parametrization)

Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) be a path such that its derivative is continuous. $\mathbf{r}(t)$ is called smooth if $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in (a, b)$, and if distinct points in (a, b) map to distinct points on the curve.

Remark.

1. According to the definition, a smooth path can be closed [i.e., $\mathbf{r}(a) = \mathbf{r}(b)$ is allowed] but cannot intersect itself, be tangent to itself, or (partly or completely) retrace itself (like the path $\mathbf{r}_2$ of **Example 1**).
2. Note that a smooth path has a nonzero velocity at every point.

Now we are ready to define the length of a (smooth) curve.

Definition. Smooth Curve and Its Length

A curve $\mathbf{r}$ is called smooth if it has a C^1 (derivative is continuous) parametrization that is smooth.

The length $\ell(\mathbf{r})$ of $\mathbf{r}$ (also called the arc-length) is defined as the length of that smooth parametrization.

That is $\ell(\mathbf{r}) = \displaystyle\int_a^b \|\mathbf{r}'(t)\| dt$.

Remarks.

- In order to show that a curve $\mathbf{r}$ is smooth, all we have to do is to find one smooth parametrization of $\mathbf{r}$.
- Its length is then equal to the length of the ~~curve.~~ path
- As a consequence of the Change of Variables Theorem (in an integral), later this semester we will show that the length of a curve is independent of the smooth parametrization that is used in the computation.

Example 2. (Related to Q3 in WebWork)

Find the length of the curve $\boldsymbol{r}(t) = \left\langle 4t, \frac{t^3}{3}, \frac{\sqrt{8}t^2}{2} \right\rangle$ for $-1 \leq t \leq 2$.

ANS: Step 1: Compute the tangent vector $\vec{r}'(t)$:

$$\vec{r}'(t) = \langle 4, t^2, \sqrt{8}t \rangle$$

Step 2. Compute the length of $\vec{r}'(t)$:

$$\|\vec{r}'(t)\| = \sqrt{4^2 + (t^2)^2 + (\sqrt{8}t)^2} = \sqrt{16 + t^4 + 8t^2}$$

$$= \sqrt{(t^2+4)^2} = t^2 + 4$$

Step 3. Compute the length of the curve

$$L(t) = \int_{-1}^{2} \|\vec{r}'(t)\| dt$$

$$= \int_{-1}^{2} (t^2 + 4)\, dt$$

$$= \left[\frac{1}{3}t^3 + 4t\right]_{-1}^{2}$$

$$= \left(\frac{1}{3} \cdot 2^3 + 4 \cdot 2\right) - \left(\frac{1}{3} \cdot (-1)^3 + 4 \times (-1)\right)$$

$$= \left(\frac{8}{3} + 8\right) - \left(-\frac{1}{3} - 4\right)$$

$$= \frac{9}{3} + 12$$

$$= \underline{15}$$

Example 3. (Related to Q2 in WebWork)

Compute the length of the curve $\mathbf{r}(t) = 3t\mathbf{i} + 8t\mathbf{j} + \left(t^2 - 8\right)\mathbf{k}$ over the interval $0 \le t \le 7$.

Hint: use the formula

$$\int \sqrt{t^2 + a^2}\,dt = \frac{1}{2}t\sqrt{t^2+a^2} + \frac{1}{2}a^2 \ln\left(t + \sqrt{t^2+a^2}\right) + C$$

ANS: Step 1. $\vec{r}'(t) = 3\vec{i} + 8\vec{j} + 2t\vec{k}$

Step 2. $\|\vec{r}'(t)\| = \sqrt{3^2 + 8^2 + (2t)^2} = \sqrt{4t^2+73}$

Step 3. $L(t) = \int_0^7 \|\vec{r}'(t)\|\,dt = \int_0^7 \sqrt{4t^2+73}\,dt$

We substitute $u = 2t$, then $du = 2dt \Rightarrow dt = \frac{1}{2}du$

When $t=0$, $u = 2\cdot 0 = 0$, when $t = 7$, $u = 2\cdot 7 = 14$

Then $L = \frac{1}{2}\int_0^{14} \sqrt{u^2 + \overset{a^2}{73}}\,du$

$$= \frac{1}{2}\left[\frac{1}{2}u\sqrt{u^2+73} + \frac{1}{2}\cdot 73 \ln\left(u + \sqrt{u^2+73}\right)\right]_0^{14}$$

$$= \frac{1}{4}\cdot 14\cdot\sqrt{14^2+73} + \frac{1}{4}\cdot 73\ln\left(14 + \sqrt{14^2+73}\right) - \frac{73}{4}\ln\sqrt{73}$$

$$= \frac{7}{2}\sqrt{269} + \frac{73}{4}\ln\frac{14+\sqrt{269}}{\sqrt{73}}$$

$$\approx 80.57$$

Example 4. (Related to Q4 in WebWork)

Find the arc-length of the curve $\mathbf{r}(t) = \langle -2\sin t, -3t, -2\cos t\rangle, -2 \le t \le 2$

ANS: Step 1. $\vec{r}\,'(t) = \langle -2\cos t, -3, 2\sin t \rangle$

Step 2. $\|\vec{r}\,'(t)\| = \sqrt{(-2\cos t)^2 + (-3)^2 + (2\sin t)^2}$

$= \sqrt{4\cos^2 t + 9 + 4\sin^2 t}$

Use the Pythagorean identity $\sin^2 t + \cos^2 t = 1$,

we get $\|\vec{r}\,'(t)\| = \sqrt{13}$

Step 3.

$$L = \int_{-2}^{2} \|\vec{r}\,'(t)\|\,dt$$

$$= \int_{-2}^{2} \sqrt{13}\,dt$$

$$= \sqrt{13}\,t \Big|_{-2}^{2}$$

$$= 4\sqrt{13}$$

Exercies 5. (Related to Q6 in WebWork)

Find the length of the curve $\boldsymbol{r}(t) = \left\langle e^{\frac{t}{9}} \cos\left(\frac{t}{9}\right), e^{\frac{t}{9}} \sin\left(\frac{t}{9}\right), e^{\frac{t}{9}} \right\rangle$ for $0 \leq t \leq 4$.

Solution.

Step 1: Compute the derivative, $\mathbf{r}'(t)$. We get

$$\mathbf{r}'(t) = \left\langle \frac{1}{9} e^{\frac{t}{9}} \cos\left(\frac{t}{9}\right) - \frac{1}{9} e^{\frac{t}{9}} \sin\left(\frac{t}{9}\right), \frac{1}{9} e^{\frac{t}{9}} \sin\left(\frac{t}{9}\right) + \frac{1}{9} e^{\frac{t}{9}} \cos\left(\frac{t}{9}\right), \frac{1}{9} e^{\frac{t}{9}} \right\rangle$$

Step 2: Compute the magnitude of the derivative, $\|\mathbf{r}'(t)\|$.

$$||\mathbf{r}'(t)\| = \sqrt{\left(\frac{1}{9} e^{\frac{t}{9}} \cos\left(\frac{t}{9}\right) - \frac{1}{9} e^{\frac{t}{9}} \sin\left(\frac{t}{9}\right)\right)^2 + \left(\frac{1}{9} e^{\frac{t}{9}} \sin\left(\frac{t}{9}\right) + \frac{1}{9} e^{\frac{t}{9}} \cos\left(\frac{t}{9}\right)\right)^2 + \left(\frac{1}{9} e^{\frac{t}{9}}\right)^2}$$

After simplifying (and using trigonometric identities to simplify the sum of squares of sine and cosine terms), we get:

$$\|\mathbf{r}'(t)\| = \frac{1}{9} e^{\frac{t}{9}} \sqrt{2+1}$$
$$\|\mathbf{r}'(t)\| = \frac{\sqrt{3} e^{\frac{t}{9}}}{9}$$

Step 3: Compute the arc-length of the curve.

The arc-length L of the curve over the interval $[0, 4]$ is:

$$\begin{aligned} L &= \int_0^4 \|\mathbf{r}'(t)\| dt \\ &= \int_0^4 \frac{\sqrt{3} e^{\frac{t}{9}}}{9} dt \\ &= \frac{\sqrt{3}}{9} \left[9 e^{\frac{t}{9}}\right]_0^4 \\ &= \sqrt{3} \left[e^{\frac{4}{9}} - 1\right] \end{aligned}$$

Definition. Arc-Length Function

Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) be a path such that its derivative is continuous. The ***arc-length function*** $s(t)$ of $\mathbf{r}(t)$ is given by

$$s(t) = \int_a^t \|\mathbf{r}'(\tau)\| d\tau$$

Remark.

- Geometrically, the arc-length function $s(t)$ measures the length of $\mathbf{r}$ from $\mathbf{r}(a)$ to the point $\mathbf{r}(t)$; that is, the distance traversed in time t.
- Clearly, $s(a) = \int_a^a \|\mathbf{r}'(\tau)\| d\tau = 0$ and $s(b) = \int_a^b \|\mathbf{r}'(\tau)\| d\tau = \ell(\mathbf{r})$.
- By the Fundamental Theorem of Calculus,

$$\frac{d}{dt} s(t) = \frac{d}{dt}\left(\int_a^t \|\mathbf{r}'(\tau)\| d\tau\right) = \|\mathbf{r}'(t)\|.$$

- $ds(t)/dt$ is the rate of change of the arc-length (i.e., the distance) with respect to time, which is the speed $\|\mathbf{r}'(t)\|$.

Curvature

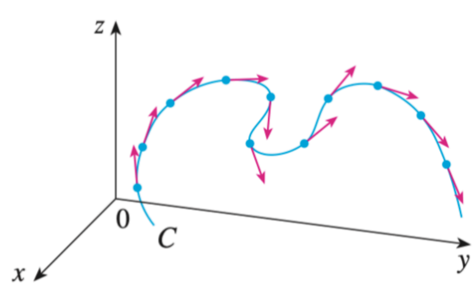
If C is a smooth curve defined by the vector function $\mathbf{r}$, then $\mathbf{r}'(t) \neq \mathbf{0}$.

The unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$

and indicates the direction of the curve.

From the following figure you can see that $\mathbf{T}(t)$ changes direction very slowly when C is relatively straight, but it changes direction more quickly when C bends or twists more sharply.



The *curvature* of C at a given point is a measure of how quickly the curve changes direction at that point.

We define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

Definition. Curvature

The *curvature* of a curve is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

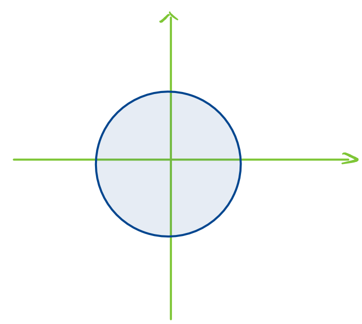
where $\mathbf{T}$ is the unit tangent vector and s is the arc-length function.

Notice that using Chain Rule, we have $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$ and $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}/dt}{ds/dt} \right\|$.

Recall $ds/dt = ||\mathbf{r}'(t)||$. So we have

$$\boxed{\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}}$$

Example 6. Show that the curvature of a circle of radius R is $1/R$.



ANS: We can take the circle have center at the origin, then its parametrization is

$$\vec{r}(t) = R\cos t\,\vec{i} + R\sin t\,\vec{j}$$

Then $\vec{r}\,'(t) = -R\sin t\,\vec{i} + R\cos t\,\vec{j}$

and $||\vec{r}\,'(t)|| = R$

So $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$\vec{T}\,'(t) = -\cos t\,\vec{i} - \sin t\,\vec{j}$

Thus $||\vec{T}\,'(t)|| = 1$ thus

$$\kappa(t) = \frac{||T'(t)||}{||\vec{r}\,'(t)||} = \frac{1}{R}$$